Global stability of neural networks with distributed delays

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In this paper, a model describing the dynamics of recurrent neural networks with distributed delays is considered. Some sufficient criteria are derived ensuring the global asymptotic stability of distributed-delay recurrent neural networks with more general signal propagation functions by introducing real parameters $p>1$, $q_{ij}>0$, and $r_{jj}>0$, $i,j=1,...,n$, and applying the properties of the M matrix and inequality techniques. We do not assume that the signal propagation functions satisfy the Lipschitz condition and do not require them to be bounded, differentiable, or strictly increasing. Moreover, the symmetry of the connection matrix is also not necessary. These criteria are independent of the delays and possess infinitely adjustable real parameters, which is important in signal processing, especially in moving image treatment and the design of networks.

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I. INTRODUCTION

In the past decade, neural networks such as Hopfield neural networks [1], cellular neural networks [2], and bidirectional associative memory networks [3,4] have attracted the attention of the scientific community (e.g., mathematicians, physicists, and computer scientists), since they have a wide range of applications, for example, pattern recognition, associative memory, and combinatorial optimization. Such applications heavily depend on the dynamical behaviors. Thus, an analysis of the dynamical behaviors is a necessary step for practical design of neural networks.

One of the most investigated problems in the dynamical behaviors of neural networks is the global asymptotic stability (GAS) of the equilibrium point. The property of GAS, which means that the domain of attraction of the equilibrium point is the whole space, is of importance from a theoretical as well as an applications viewpoint in several fields [5,6]. In particular, in the neural field, GAS networks are known to be well suited for solving some classes of optimization problems in real time [7–10], with connections to adaptive control also [10]. In fact, a GAS neural network is guaranteed to compute the global optimal solution independently of the initial conditions, which in turn implies that the network is devoid of spurious suboptimal responses. Such GAS neural circuits can also be useful for accomplishing other interesting cognitive or computational tasks [11,12]. Thus, many scientific and technical workers have been joining the study field with great interest, and various interesting results on the GAS of neural networks with constant delays or without delays have been reported [13–17]. As is well known, the use of constant fixed delays in models of delayed feedback provides a good approximation in simple circuits consisting of a small number of cells. However, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. In these circumstances, the signal propagation is not instantaneous and cannot be modeled with discrete delays, and a more appropriate way is to incorporate continuously distributed delays. Recently, some authors (Gopalsamy and He [18], Zhang and Jin [19]) have studied the GAS of Hopfield neural networks with distributed delays. In [18], Gopalsamy and He considered the following system of integro-differential equations as a model for Hopfield neural networks with continuously distributed delays:

$$\dot{x}_i(t)=-a_i x_i(t)+\sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} k_{ij}(t-s)g_j(x_j(s))ds+I_i,\quad t\geqslant 0, \tag{1}$$

$$x_i(t)=\phi_i(t),\quad -\infty<t\leqslant 0,\quad i=1,...,n,$$

where $a_i>0$ represent the passive decay rates, $i=1,...,n$, in which n corresponds to the number of units in the network, $x_i(t)$ correspond to the state vectors of the ith neural unit at time t, b_{ij} are the synaptic connection strengths, g_j are the signal propagation functions, I_i are the constant inputs from outside the system, and ϕ_i are assumed to be bounded and continuous functions on $(-\infty,0]$. The kernel functions $k_{ij}:[0,+\infty)\to[0,+\infty)$ $(i,j=1,...,n)$ are continuous on $[0,+\infty)$ with $\int_0^{+\infty}k_{ij}(s)ds=1$, and satisfy

$$\int_0^{+\infty} s k_{ij}(s)ds<+\infty,\quad i,j=1,...,n. \tag{2}$$

These authors derived the criteria of GAS for the system (1) when the hypothesis (2) held.

In [19], Zhang and Jin also discussed the model (1), and gave the criteria for GAS provided that the hypothesis (2) held and g_j satisfied the Lipschitz conditions. However, it has become increasingly clear that hypothesis (2) has imposed serious constraint on both physical realization and practical application of the networks. On the other hand, there are many signal propagation functions which do not satisfy the global Lipschitz condition. Kosko [20] and Feng and Plamondon [21] describes several commonly used functions that are not Lipschitz.

Motivated by the above discussion, in this paper, we consider the global asymptotic stability of a class of distributed-delay recurrent neural networks with general functions, which are described by the following system of integro-differential equations:

$$\dot{x}_i(t) = -h_i(x_i(t)) + \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} k_{ij}(t-s) g_j(x_j(s)) ds + I_i, \quad t \geqslant 0, \tag{3}$$

$$x_i(t) = \phi_i(t), \quad -\infty < t \leqslant 0, \quad i = 1,...,n,$$

where $h_i(x_i(t))$ is continuous and differentiable.

It can be easily seen that the recurrent neural networks described by system (3) are an extension of system (1), and include Hopfield neural networks and BAM (bidirectional associative memory) networks. To the best of the author's knowledge, few authors [22] have considered the global asymptotic stability of the recurrent neural networks (3). Our objective in this paper is to study GAS for the distributed-delay recurrent neural networks (3) and to give a set of criteria ensuring global asymptotic stability of this system with more general signal propagation functions by introducing real parameters $p>1$, $q_{ij}>0$, and $r_{jj}>0$, $i,j=1,...,n$, and applying the properties of the M matrix and inequality techniques. The results related in Refs. [18], [19], [22] and the references cited therein are extended and improved. Moreover, these criteria are independent of delays and possess infinitely adjustable real parameters, and they are easy to check and apply in practice; which is of prime importance and great interest in many application fields and the design of networks. Here, we point out that our methods, which are different from previously known results, are based on the properties of the M matrix [23,24] and inequality techniques [3,25–27]. Our theorems drop the Lipschitz condition, assumption (2), and do not require the signal propagation functions to be differentiable, bounded, or monotonically increasing.

II. PRELIMINARY THEORY

Let $C[X,Y]$ be a continuous mapping set from the topological space X to the topological space Y, and $R_+ = [0, +\infty)$. In particular, $C \triangleq C[(-\infty,0],R^n]$.

For any $\phi = (\phi_1,\ldots,\phi_n)^T \in C$, a solution of the system (3) is a function $x = (x_1,\ldots,x_n)^T$: $R_+ \to R^n$ satisfying Eq. (3) for $t \geqslant 0$. Throughout the paper, we always assume that the system (3) has a continuous solution denoted by $x(t,0,\phi)$ or simply $x(t)$ if no confusion should occur. $A \geqslant B$ $(A<B)$ means that each pair of corresponding elements of A and B satisfies the inequality $\geqslant$ $(<)$. In particular, A is called a non-negative vector if $A \geqslant 0$.

For $x \in R^n$, we define $[x(t)]^+ = (|x_1(t)|,\ldots,|x_n(t)|)^T$. For any $\phi \in C$, $[\phi]_\infty^+ = (\|\phi_1\|_\infty,\ldots,\|\phi_n\|_\infty)^T$, where $\|\phi_i\|_\infty = \sup_{-\infty<s\leqslant 0}|\phi_i(s)|$, $i=1,...,n$.

Definition 1. $x(t) = x^* \in R^n$ is called an equilibrium point of Eq. (3), if the constant vector $x^* = (x_1^*,\ldots,x_n^*)^T$ satisfies

$$h_i(x_i^*) = \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} k_{ij}(t-s) g_j(x_j^*) ds + I_i \tag{4}$$

for $i=1,...,n$.

Definition 2. A real $n\times n$ matrix $A=(a_{ij})$ is said to be an M matrix if $a_{ij} \leqslant 0$ for $i,j=1,...,n$ and $i \neq j$, and all successive principal minors of A are positive.

Definition 3. If $f(t)$: $R\to R$ is a continuous function, then $D^+f(t)/dt$ is defined as

$$\frac{D^+f(t)}{dt} = \lim_{h\to 0^+}\frac{1}{h}[f(t+h)-f(t)].$$

Lemma 1. Let $A=(a_{ij})$ be an $n\times n$ matrix with nonpositive off-diagonal elements [23,24], then the following statements are true.

(1) A is an M matrix if and only if the real parts of all eigenvalues of A are positive.

(2) A is an M matrix if and only if there exists a vector $\xi > 0$ such that $\xi^T A > 0$.

(3) A is an M matrix if and only if there exists $w_j>0$ $(j=1,...,n)$, such that

$$\sum_{j=1}^{n} a_{ij} w_j > 0, \quad i=1,...,n.$$

(4) A is an M matrix if and only if there exists $w_i>0$ $(i=1,...,n)$, such that

$$\sum_{i=1}^{n} a_{ij} w_i > 0, \quad j=1,...,n.$$

III. GLOBAL ASYMPTOTIC STABILITY

Let $x^* = (x_1^*,\ldots,x_n^*)^T$ be an equilibrium point of the system (3). Denoting $y_i(t) = x_i(t) - x_i^*$, for each $i=1,...,n$, then the system (3) can be rewritten as follows:

$$\dot{y}_i(t) = -\tilde{h}_i(y_i(t)) + \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} k_{ij}(t-s) f_j(y_j(s)) ds, \quad t \geqslant 0, \tag{5}$$

$$y_i(t) = \psi_i(t), \quad -\infty < t \leqslant 0, \quad i=1,...,n,$$

in which $f_j(y_j) = g_j(x_j) - g_j(x_j^*)$, $\psi_i(t) = \phi_i(t) - x_i^*$, $\tilde{h}_i(y_i) = h_i(x_i) - h_i(x_i^*)$, $y(t) = (y_1(t),\ldots,y_n(t))^T$, $\psi(t) = (\psi_1(t),\ldots,\psi_n(t))^T$.

Clearly, x^* for Eq. (3) is uniformly stable (US) and GAS if and only if the equilibrium point O of Eq. (5) is US and GAS, respectively.

Throughout the paper, we always assume the following.

Hypothesis H_1. h_i are differentiable, $a_i \triangleq \inf_{x_i\in R}\{\dot{h}_i(x_i)\} > 0$, and $h_i(0)=0$ $(i=1,...,n)$, where $\dot{h}_i(x_i)$ represents the derivative of $h_i(x_i)$.

Hypothesis H_2. The functions f_j $(j=1,...,n)$ satisfy $y_j f_j(y_j) > 0$ $(y_j \neq 0)$, and there exist positive constants λ_j $(j=1,...,n)$ such that

$$\lambda_j = \sup_{y_j \neq 0} \frac{f_j(y_j)}{y_j}, \quad \forall y_j \in R. \tag{6}$$

Hypothesis H_3. There exist constants q_{ij}, $r_{jj} \in R$, and $p > 1$, such that $A-(B\Lambda+\tilde{B}\tilde{\Lambda})$ is an M matrix, where $A = \text{diag}\{a_i\}$, $B = ((p-1)/p|b_{ij}|^{(p-q_{ij})/(p-1)})$, $\Lambda = \text{diag}\{\lambda_j^{(p-r_{jj})/(p-1)}\}$, $\tilde{B} = ((1/p)|b_{ij}|^{q_{ij}})$, $\tilde{\Lambda} = \text{diag}\{\lambda_j^{r_{jj}}\}$, $i,j = 1,...,n$.

Theorem 1. For system (5), suppose that h_i and f_i satisfy the hypotheses H_1 and H_2 above. Assume furthermore that the parameters of system (5) satisfy H_3. Then the equilibrium point O of Eq. (5) is US.

Proof. Since $A-(B\Lambda+\tilde{B}\tilde{\Lambda})$ is an M matrix, by Lemma 1, there exist constants $w_j > 0$ $(j = 1,...,n)$, such that

$$a_i^{-1} p^{-1} \left\{ \sum_{j=1}^{n} w_i^{-1} w_j [(p-1)\lambda_j^{(p-r_{jj})/(p-1)} |b_{ij}|^{(p-q_{ij})/(p-1)} + \lambda_j^{r_{jj}} |b_{ij}|^{q_{ij}}] \triangleq \gamma_i < 1, \right. \tag{7}$$

where $i,j = 1,...,n$.

Let

$$z_i(t) = \begin{cases} w_i^{-1} y_i(t), & t \geq 0, \\ w_i^{-1} \psi_i(t) \triangleq \varphi_i(t), & -\infty < t \leq 0. \end{cases} \tag{8}$$

Then, the system (5) can be rewritten as

$$\dot{z}_i(t) = -w_i^{-1} \tilde{h}_i(w_i z_i(t)) + w_i^{-1} \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_{ij}(t-s) \times f_j(w_j z_j(s)) ds, \quad t \geq 0,$$

$$z_i(t) = \varphi_i(t), \quad -\infty < t \leq 0, \quad i = 1,...,n. \tag{9}$$

Obviously, the equilibrium point O of system (9) is US and GAS if and only if the equilibrium point O of system (5) is US and GAS, respectively. We now show for any $\epsilon > 0$ and $\varphi = (\varphi_1, \ldots, \varphi_n)^T \in C$, when $[\varphi]_\infty^+ < E\epsilon$,

$$[z(t)]^+ < E\epsilon \quad \text{for } t \geq 0, \tag{10}$$

where $[z(t)]^+ = (|z_1(t)|, \ldots, |z_n(t)|)^T$, $E = (1,...,1)^T$.

If (10) is false, then there must be some i and $t_1 > 0$ such that

$$|z_i(t_1)| = \epsilon \tag{11}$$

and

$$[z(t)]^+ \leq E\epsilon \quad \text{for } t \leq t_1. \tag{12}$$

From Eq. (9), we have

$$\begin{aligned} \frac{D^+|z_i(t)|}{dt} &= \text{sgn}\, z_i(t) \dot{z}_i(t) \\ &= \text{sgn}\, z_i(t) \left(-w_i^{-1} \tilde{h}_i(w_i z_i(t)) + w_i^{-1} \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_{ij}(t-s) f_j(w_j z_j(s)) ds \right) \\ &\leq -a_i |z_i(t)| + \sum_{j=1}^{n} w_i^{-1} w_j |b_{ij}| \int_{-\infty}^{t} k_{ij}(t-s) \lambda_j |z_j(s)| ds. \end{aligned} \tag{13}$$

So

$$\begin{aligned} |z_i(t_1)| &\leq e^{-a_i t_1} |z_i(0)| + \int_0^{t_1} e^{-a_i(t_1-s)} \left(\int_{-\infty}^{s} \sum_{j=1}^{n} w_i^{-1} w_j |b_{ij}| \lambda_j k_{ij}(s-\theta) |z_j(\theta)| d\theta \right) ds \\ &= e^{-a_i t_1} |z_i(0)| + \int_0^{t_1} e^{-a_i(t_1-s)} \left(\int_{-\infty}^{s} \sum_{j=1}^{n} w_i^{-1} w_j [\lambda_j^{(p-r_{jj})/(p-1)} |b_{ij}|^{(p-q_{ij})/(p-1)}]^{(p-1)/p} [\lambda_j^{r_{jj}} |b_{ij}|^{q_{ij}}]^{1/p} \right. \\ &\quad \left. \times k_{ij}(s-\theta) |z_j(\theta)| d\theta \right) ds. \end{aligned}$$

By using the inequality $a^k b^{1-k} \leq ka + (1-k)b$ for $0 < k < 1$ and $a,b > 0$, we obtain an estimate for the right-hand side of the inequality above:

$$|z_i(t_1)| \leqslant \|\varphi_i\|_\infty e^{-a_i t_1} + \int_0^{t_1} e^{-a_i(t_1-s)} \left(\int_{-\infty}^{s} p^{-1} \sum_{j=1}^{n} w_i^{-1} w_j [(p-1)\lambda_j^{(p-r_{jj})/(p-1)} |b_{ij}|^{(p-q_{ij})/(p-1)} + \lambda_j^{r_{jj}} |b_{ij}|^{q_{ij}}] \times k_{ij}(s-\theta)|z_j(\theta)| d\theta \right) ds$$

$$\leqslant e^{-a_i t_1}\epsilon + \int_0^{t_1} e^{-a_i(t_1-s)} \left(\int_0^{+\infty} p^{-1} \sum_{j=1}^{n} w_i^{-1} w_j [(p-1)\lambda_j^{(p-r_{jj})/(p-1)} |b_{ij}|^{(p-q_{ij})/(p-1)} + \lambda_j^{r_{jj}} |b_{ij}|^{q_{ij}}] k_{ij}(\theta)\epsilon\, d\theta \right) ds$$

$$< e^{-a_i t_1}\epsilon + \int_0^{t_1} e^{-a_i(t_1-s)} \left(\int_0^{+\infty} k_{ij}(\theta) a_i \epsilon\, d\theta \right) ds = e^{-a_i t_1}\epsilon + (1-e^{-a_i t_1})\epsilon = \epsilon,$$

which contradicts the equality (11), and so (10) holds. By the meaning of uniform stability, we derive that the equilibrium point O of system (9) is US.

By the proof of the above theorem and Eq. (10), for any given $\varphi \in C$, there must be some $\epsilon_0 > 0$. When $[\varphi]_\infty^+ < E\epsilon_0$, we have a solution $z(t)$ of the system (9) that satisfies

$$[z(t)]^+ < E\epsilon_0 \quad \text{for } t \geqslant 0,$$

which leads to the following corollary.

Corollary 1. If the conditions H_1–H_3 hold, then the solutions of system (9) are uniformly bounded.

Theorem 2. Suppose that H_1–H_3 hold, then the equilibrium point O of system (9) is globally attractive, which means the equilibrium point O of system (5) is also globally attractive.

Proof. For any given $\varphi \in C$, we first prove

$$\limsup_{t\to+\infty} [z(t)]^+ = 0, \tag{14}$$

where $\limsup_{t\to+\infty}[z(t)]^+ = (\limsup_{t\to+\infty}|z_1(t)|, \ldots, \limsup_{t\to+\infty}|z_n(t)|)^T$. From Corollary 1, there exists a non-negative constant vector $\sigma = (\sigma_1, \ldots, \sigma_n)^T$, such that

$$\limsup_{t\to+\infty} [z(t)]^+ = \sigma. \tag{15}$$

According to the definition of the superior limit and Corollary 1, for a sufficiently small constant $\varepsilon > 0$, there is $t_2 > 0$, such that

$$[z(t)]^+ \leqslant (1+\epsilon)\sigma \quad \text{for any } t \geqslant t_2. \tag{16}$$

Since $\int_0^\infty k_{ij}(s)ds = 1$, $i,j = 1,...,n$, for the above ε there must be $T > 0$ such that

$$\gamma_i \epsilon_0 \int_T^\infty k_{ij}(s) ds \leqslant \varepsilon, \quad i = 1,...,n. \tag{17}$$

From Eqs. (13), (16), and (17), when $t \geqslant t_2 + T$, we obtain

$$\frac{D^+|z_i(t)|}{dt} + a_i|z_i(t)| \leqslant \int_{-\infty}^{t} \sum_{j=1}^{n} w_i^{-1} w_j |b_{ij}| \lambda_j k_{ij}(t-s)|z_j(s)| ds = \left(\int_{-\infty}^{t-T} + \int_{t-T}^{t} \right) \sum_{j=1}^{n} w_i^{-1} w_j |b_{ij}| \lambda_j k_{ij}(t-s)|z_j(s)| ds$$

$$= \int_{-\infty}^{t-T} \sum_{j=1}^{n} w_i^{-1} w_j |b_{ij}| \lambda_j k_{ij}(t-s)|z_j(s)| ds + \int_{t-T}^{t} \sum_{j=1}^{n} w_i^{-1} w_j |b_{ij}| \lambda_j k_{ij}(t-s)|z_j(s)| ds$$

$$= \int_{-\infty}^{t-T} \sum_{j=1}^{n} w_i^{-1} w_j [\lambda_j^{(p-r_{jj})/(p-1)} |b_{ij}|^{(p-q_{ij})/(p-1)}]^{(p-1)/p} [\lambda_j^{r_{jj}} |b_{ij}|^{q_{ij}}]^{1/p} k_{ij}(t-s)|z_j(s)| ds$$

$$+ \int_{t-T}^{t} \sum_{j=1}^{n} w_i^{-1} w_j [\lambda_j^{(p-r_{jj})/(p-1)} |b_{ij}|^{(p-q_{ij})/(p-1)}]^{(p-1)/p} [\lambda_j^{r_{jj}} |b_{ij}|^{q_{ij}}]^{1/p} k_{ij}(t-s)|z_j(s)| ds.$$

Again, by using the inequality $a^k b^{1-k} \leqslant ka + (1-k)b$ for $0 < k < 1$ and $a,b > 0$, we obtain an estimate for the right-hand side of the inequality above:

$$\frac{D^+|z_i(t)|}{dt}+a_i|z_i(t)|\leq p^{-1}\int_{-\infty}^{t-T}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}]k_{ij}(t-s)|z_j(s)|ds$$
$$+p^{-1}\int_{t-T}^{t}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}]k_{ij}(t-s)|z_j(s)|ds$$
$$\leq p^{-1}\int_{T}^{+\infty}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}]k_{ij}(s)\epsilon_0 ds$$
$$+p^{-1}\sum_{j=1}^{n} w_i^{-1}w_i[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}](1+\varepsilon)\sigma_j$$
$$\leq a_i\varepsilon+P^{-1}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}](1+\varepsilon)\sigma_j .$$

It follows from the above inequality that we have

$$|z_i(t)|\leq e^{-a_it}|z_i(0)|+\int_0^t e^{-a_i(t-s)}\left(a_i\varepsilon+p^{-1}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}](1+\varepsilon)\sigma_j ds\right)$$
$$\leq e^{-a_it}|z_i(0)|+(1-e^{-a_it})\left(\varepsilon+a_i^{-1}p^{-1}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}](1+\varepsilon)\sigma_j\right).$$

Combining Eq. (15) with the definition of the superior limit, there are $t_l\geq t_2+T$, $l=1,2,...$, such that $\lim_{t_l\to+\infty}|z_i(t_l)|=\sigma_i$, $i=1,...,n$.

Letting $t_l\to+\infty$, $\varepsilon\to 0$, so we have

$$\sigma_i\leq a_i^{-1}p^{-1}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}\times|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}]\sigma_j$$
$$\leq a_i^{-1}p^{-1}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}\times|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}]\alpha,$$

where $i=1,2,...,n$ and $\alpha\triangleq\max_j\{\sigma_j\}$. Thus, we obtain

$$\alpha\leq\max_i\left(a_i^{-1}p^{-1}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}\times|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}]\right)\alpha.$$

If $\sigma\neq 0$, then α must be a positive constant. That is,

$$\max_i\left(a_i^{-1}p^{-1}\sum_{j=1}^{n} w_i^{-1}w_j[(p-1)\lambda_j^{(p-r_{jj})/(p-1)}\times|b_{ij}|^{(p-q_{ij})/(p-1)}+\lambda_j^{r_{jj}}|b_{ij}|^{q_{ij}}]\right)\geq 1,$$

which contradicts the condition H_3. Hence, σ must be a zero vector and Eq. (14) holds. Therefore, we obtain $\lim_{t\to+\infty}[z(t)]^+=0$, i.e., $\lim_{t\to+\infty}[y(t)]^+=0$. This completes the proof.

By Theorems 1 and 2, we obtain the following result.

Theorem 3. Assume that the conditions H_1–H_3 hold, then the equilibrium point x^* of system (3) is GAS.

Corollary 2. If the signal propagation functions g_j are globally Lipschitz with Lipschitz constants λ_j, i.e., $|g_j(u_j)-g_j(v_j)|\leq\lambda_j|u_j-v_j|$ for all $u_j,v_j\in R$. Furthermore, we assume that the conditions H_1 and H_3 hold. Then the equilibrium point x^* of system (3) is also GAS.

Proof. Clearly, the globally Lipschitz conditions imply H_2. From Theorem 3, we obtain that the equilibrium point x^* of system (3) is GAS.

To compare Theorem 3 with some previous results in [18,19,22], we make the following remarks.

Remark 1. Let $p=2$, $r_{jj}=q_{ij}=1$ in H_3, and $h_i(x_i)=a_ix_i$, respectively. Taking the signal propagation functions $g_j(x_j)=\tanh(\lambda_jx_j)$, $i,j=1,...,n$. Moreover, if the condition H_3 and hypothesis (2) hold, then we can easily obtain the result given in Gopalsamy and He [18] as a special case.

Remark 2. Let $p=2$, $r_{jj}=q_{ij}=1$ $(i,j=1,...,n)$ in H_3, respectively. Then the condition H_3 becomes the following form:

$$A-H\Lambda \quad \text{is an } M \text{ matrix,}$$

where $H=B+\tilde{B}=(|b_{ij}|)$ and $\Lambda=\tilde{\Lambda}=\mathrm{diag}\{\lambda_i\}$. By Definition 1, we easily obtain that $A-H\Lambda$ is an M matrix if and only if $A\Lambda^{-1}-H$ is an M matrix.

If hypothesis (2) holds, the signal propagation functions $g_j(x_j)$ satisfy globally Lipschitz conditions with Lipschitz constants λ_j, and $A\Lambda^{-1}-H$ is an M matrix. Assume furthermore that H_1 is satisfied. Then we can easily obtain the result given in [22] as a corollary.

Remark 3. The result of Zhang and Jin in [19] stated that system (3) with $h_i(x_i)=a_i x_i$ was GAS provided hypothesis (2) held, the signal propagation functions $g_j(x_j)$ satisfied globally Lipschitz conditions with Lipschitz constants λ_j, and $-(A\Lambda^{-1}+H)$ was an M matrix. Here, we point out that the statement "$-(A\Lambda^{-1}+H)$ was an M matrix" in [19] is not true. From the proof of the theorem in [19], the condition should be that $A\Lambda^{-1}-H$ is an M matrix. Moreover, the global Lipschitz conditions and hypothesis (2) are not necessary in our results.

IV. ILLUSTRATIVE EXAMPLES

Example 1. Let $n=1$, $h_1(x_1)\equiv h(x)=2[x(t)+e^{x(t)}-1]$, $g_1(x_1)\equiv g(x)=x$, and $k_{11}(t-s)=2/\pi[1+(t-s)^2]$. Obviously, $\int_0^\infty k_{11}(s)ds=1$, $a_1=\inf_{x\in R}\dot{h}_1(x)=2$, and $\lambda_1=1$. Consider the scalar recurrent neural network system

$$\dot{x}(t)=-2[x(t)+e^{x(t)}-1]+\int_{-\infty}^{t}\frac{2}{\pi[1+(t-s)^2]}x(s)ds. \tag{18}$$

Taking $b_{11}=1$, $p=2$, $q_{11}=r_{11}=1$. By simple calculation, we obtain that $A-(B\Lambda+\tilde{B}\tilde{\Lambda})=1$ is a 1×1 M matrix, and $x^*=0$ is an equilibrium point of Eq. (18). In view of Theorem 3, we derive that $x^*=0$ is GAS. It should be noted here that $\int_0^\infty sk_{11}(s)ds=\infty$, i.e., hypothesis (2) is not satisfied. However, the GAS of the equilibrium point for system (18) cannot be derived by using the methods of [18,19,22].

Example 2. Consider the continuously-distributed-delay recurrent neutral network system

$$\dot{x}_1(t)=-h_1(x_1(t))+\sum_{j=1}^{2} b_{1j}\int_{-\infty}^{t}k_{1j}(t-s)g_j(x_j(s))ds+I_1, \tag{19}$$

$$\dot{x}_2(t)=-h_2(x_2(t))+\sum_{j=1}^{2} b_{2j}\int_{-\infty}^{t}k_{2j}(t-s)g_j(x_j(s))ds+I_2.$$

Let $g_j(x_j)=x_j^{1/3}$, $k_{ij}(t)=2/\pi(1+t^2)$, and $h_i(x_i)=3(x_i+e^{x_i}-1)$, $i,j=1,2$. It is very easy to see that $\int_0^\infty k_{ij}(s)ds=1$, $\int_0^\infty sk_{ij}(s)ds=+\infty$, and $a_i=\inf_{x_i\in R}\dot{h}_i(x_i)=3$, $i,j=1,2$. Moreover, if we set $I_1=3e-3/4$, $I_2=3e-3/2$, $b_{11}=1/2$, $b_{12}=1/4$, $b_{21}=1/2$, and $b_{22}=1$, then Eq. (19) has an equilibrium point $x^*=(x_1^*,x_2^*)^T=(1,1)^T$ and $[g_j(x_j)-g_j(1)](x_j-1)=(x_j^{1/3}-1)(x_j-1)>0$ $(x_j\neq 1)$, that is, $f_j(y_j)y_j>0$ $(y_j\neq 0)$, $j=1,2$. However, $g_j(x_j)=x_j^{1/3}$ do not satisfy global Lipschitz conditions. In fact, if $g_j(x_j)$ are globally Lipschitz with Lipschitz constants $L_j>0$ $(j=1,2)$, i.e., there are constants $L_j>0$ such that

$$|g_j(u_j)-g_j(v_j)|\leqslant L_j|u_j-v_j|$$

$$\text{for arbitrary } u_j, v_j\in R, j=1,2.$$

Then we choose $u_j=(1/8)L_j^{-3/2}$ and $v_j=0$, $j=1,2$. Substitute u_j and v_j into the above inequality, we obtain $(1/2)L_j^{-1/2}\leqslant(1/8)L_j^{-1/2}$, that is, $1/2\leqslant 1/8$, which contradicts $1/2>1/8$. Therefore, the signal propagation functions $g_j(x_j)$ do not satisfy global Lipschitz conditions.

By employing the method of computation of the extreme value of functions, we derive $\max\{[g_j(x_j)-1]/(x_j-1)\}=4/3$ as $x_j=-1/8$, which implies $\sup_{x_j\neq 1}\{[g_j(x_j)-1]/(x_j-1)\}=4/3$, for arbitrary $x_j\in R$, i.e., $\sup_{y_j\neq 0}[f_j(y_j)/y_j]=4/3$, for arbitrary $y_j\in R$. So, the functions $f_j(y_j)$ satisfy H_2 with $\lambda_j=4/3$.

Taking $p=2$, $q_{ij}=r_{jj}=1$ $(i,j=1,2)$ in H_3, and by using Lemma 1, we obtain $A-(B\Lambda+\tilde{B}\tilde{\Lambda})=\begin{pmatrix}7/3 & -1/3\\ -2/3 & 5/3\end{pmatrix}$ is an M matrix. In view of Theorem 3, the equilibrium point $x^*=(x_1^*,x_2^*)^T=(1,1)^T$ of Eq. (19) is GAS. However, it is very difficult to obtain the result by using the techniques of [18,19,22].

V. CONCLUSIONS

In this paper, we derive some simple sufficient criteria ensuring the global asymptotic stability of distributed-delay recurrent neural networks with more general signal propagation functions by employing the properties of the M matrix and inequality techniques. These criteria are independent of delays and posses infinitely adjustable real parameters $p>1$, $q_{ij}>0$, and $r_{jj}>0$, $i,j=1,\ldots,n$. The results presented here are more general and easier to check than those given in the related literature because the restrictions of sufficient conditions are less restrictive than those in [18,19,22]. For instance, we do not assume that the signal propagation functions g_j satisfy the Lipschitz condition. Moreover, the hypothesis (2) and the symmetry of the connection matrix b_{ij} are also not necessary. For this reason, our results possess highly important significance in some applied fields, for example, the global optimization problem and the design of networks. In addition, the methods of this paper may be extended to discuss more complicated systems such as Hopfield neural networks, cellular neural networks, and bidirectional associative memory networks.

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